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On the construction of a pseudo-Hermitian quantum system with a pre-determined metric in the Hilbert space

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Abstract

A class of pseudo-Hermitian quantum system with an explicit form of the positive-definite metric in the Hilbert space is presented. The general method involves a realization of the basic canonical commutation relations defining the quantum system in terms of operators that are Hermitian with respect to a pre-determined positive-definite metric in the Hilbert space. Appropriate combinations of these operators result in a large number of pseudo-Hermitian quantum systems admitting entirely real spectra and unitary time evolution. The examples considered include simple harmonic oscillators with complex angular frequencies, Stark (Zeeman) effect with non-Hermitian interaction, non-Hermitian general quadratic form of N boson (fermion) operators, symmetric and asymmetric *XXZ* spin chain in the complex magnetic field, non-Hermitian Haldane–Shastry spin chain and Lipkin–Meshkov–Glick model.

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1. Introduction

In standard quantum physics, an operator that is equal to its own complex-conjugate transpose is defined as Hermitian. Operators not satisfying the above criteria are termed non-Hermitian and have been used extensively to stimulate dissipative quantum processes. However, the discovery [1–5] of a class of such non-Hermitian Hamiltonian admitting entirely real spectra with unitary time evolution has given a scope to review this standard practice and broaden our understanding of quantum physics [1–22]. The reality of the entire spectra is related to an underlying unbroken combined parity ($\mathcal{P}$) and time-reversal

($\mathcal{T}$) symmetry and/or pseudo-hermiticity of the non-Hermitian Hamiltonian with a positive-definite metric in the Hilbert space [1–4]. Apart from a very few known examples, one of the major technical difficulties in the study of $\mathcal{PT}$-symmetric and/or pseudo-Hermitian quantum physics is to find the appropriate basis with respect to which the non-Hermitian Hamiltonian becomes Hermitian. It may be mentioned here that the description of a pseudo-Hermitian Hamiltonian is incomplete in the absence of an explicit knowledge of the metric in the Hilbert space, since neither the completeness of states nor the unitarity can be guaranteed. There are known methods based on spectral decomposition [13, 16], perturbation theory [14, 17], Moyal product [15], group theory [10], etc that find an exact or approximate form of the metric in the Hilbert space of a given $\mathcal{PT}$-symmetric or pseudo-Hermitian Hamiltonian. However, the list is still not exhaustive and there are always scopes for introducing alternative methods and beautiful models based on inherent simplicity and physical relevance.

The purpose of this paper is to present a class of pseudo-Hermitian Hamiltonian with an explicit form of the metric in the Hilbert space. The examples include simple harmonic oscillators with complex angular frequencies, Stark (Zeeman) effect with non-Hermitian interaction, non-Hermitian general quadratic form of N boson (fermion) operators, XXZ spin chains with a complex magnetic field, non-Hermitian Haldane–Shastry spin chain [23] and Lipkin–Meshkov–Glick (LMG) [24] model. A non-Hermitian asymmetric *XXZ* spin Hamiltonian that is generally used as the time evolution operator of certain reaction–diffusion processes and growth phenomenon is shown to be pseudo-Hermitian and may thus be used to describe non-dissipative processes by using a modified inner product in the Hilbert space. The approach used in this paper is the following. The basic canonical commutation relations defining a quantum system are realized in terms of operators that are Hermitian with respect to a pre-determined positive-definite metric η_+ in the Hilbert space. Consequently, any Hamiltonian that is constructed using appropriate combination of these operators is Hermitian with respect to η_+. However, in general, the same Hamiltonian may not be Hermitian with respect to the standard Dirac-hermiticity condition, thereby giving rise to a pseudo-Hermitian Hamiltonian.

This paper is organized as follows. In the beginning of section 2, known results on pseudo-Hermitian operators [2–4] are reviewed. Thereafter, a general prescription to construct a pseudo-Hermitian quantum system with a pre-determined metric is presented. In section 3, examples of single-particle pseudo-Hermitian quantum systems are given. A two-dimensional pseudo-Hermitian simple harmonic oscillator, Stark and Zeeman effect with non-Hermitian interaction are discussed in sections 3.1, 3.2 and 3.3, respectively. Section 4 contains examples of many-particle pseudo-Hermitian quantum systems. In particular, general quadratic forms of boson and fermion operators are discussed in sections 4.1 and 4.2, respectively. In section 4.1, Schwinger's oscillator model of angular momentum is generalized to pseudo-Hermitian operators and a non-Hermitian version of the Lipkin–Meshkov–Glick model is introduced. The pseudo-Hermitian *XXZ* spin chain and Haldane–Shastry spin chain are presented in section 4.3. Finally, the findings of this paper are summarized with possible implications in section 5.

2. Formalism

An operator $\hat{A}$ that is related to its adjoint $\hat{A}^\dagger$ through a similarity transformation is known as pseudo-Hermitian [2, 3]:

$$\hat{A}^\dagger = \eta \hat{A} \eta^{-1}. \tag{1}$$

In general, the operator η is not unique for a given pseudo-Hermitian operator $\hat{A}$. Among all possible forms of η, a positive-definite η_+ is chosen to define a modified inner product in the Hilbert space of $\hat{A}$ as follows:

$$\langle\langle \cdot, \cdot \rangle\rangle_{\eta_+} := \langle \cdot, \eta_+ \cdot \rangle. \tag{2}$$

The operator η_+ plays the role of a metric in the Hilbert space and the standard inner product $\langle \cdot, \cdot \rangle$ is obtained in the limit when η_+ is replaced by the identity operator. The Hilbert space that is endowed with the metric η_+ with the modified inner product (2) is denoted as $\mathcal{H}_{\eta_+}$. On the other hand, the Hilbert space that is endowed with the standard inner product $\langle \cdot, \cdot \rangle$ is denoted as $\mathcal{H}_D$. The subscript D indicates that the Dirac-hermiticity condition is used in this Hilbert space. The pseudo-Hermitian operator $\hat{A}$ is Hermitian in the Hilbert space $\mathcal{H}_{\eta_+}$. In an alternative formulation of the same problem, $\hat{A}$ can be mapped to an operator $\hat{\mathcal{A}}$ that is Hermitian in $\mathcal{H}_D$. In particular,

$$\hat{\mathcal{A}} = \rho \hat{A} \rho^{-1}, \qquad \rho := \sqrt{\eta_+}. \tag{3}$$

The operator $\hat{\mathcal{A}}$ satisfying the above relation is known as quasi-Hermitian [4]. It may be noted that the Hilbert spaces of $\hat{A}$ and $\hat{\mathcal{A}}$ are different. Corresponding to a Hermitian operator $\hat{\mathcal{B}}$ in the Hilbert space $\mathcal{H}_D$ of $\hat{\mathcal{A}}$, a Hermitian operator $\hat{B}$ in the Hilbert space $\mathcal{H}_{\eta_+}$ of $\hat{A}$ can be defined as [2]

$$\hat{B} = \rho^{-1} \hat{\mathcal{B}} \rho. \tag{4}$$

The above relation is important for the identification of physical observables in the Hilbert space $\mathcal{H}_{\eta_+}$ of $\hat{A}$. An interesting consequence of equation (4) is that a set of operators $\hat{\mathcal{B}}_i$ obey the same canonical commutation relations as those satisfied by the corresponding set of operators $\hat{B}_i$ and vice versa.

The coordinates and the conjugate momenta which are Hermitian in the Hilbert space $\mathcal{H}_D$ that is endowed with the standard inner product $\langle \cdot, \cdot \rangle$ are denoted as (x, y, z) and (p_x, p_y, p_z), respectively. In the coordinate space representation, the momenta and the orbital angular momentum operators have the following standard form:

$$\begin{aligned} &p_x = -\mathrm{i}\frac{\partial}{\partial x}, \qquad p_y = -\mathrm{i}\frac{\partial}{\partial y}, \qquad p_z = -\mathrm{i}\frac{\partial}{\partial z}, \\ &\mathcal{L}_x = y p_z - z p_y, \qquad \mathcal{L}_y = z p_x - x p_z, \qquad \mathcal{L}_z = x p_y - y p_x. \end{aligned} \tag{5}$$

A positive-definite metric η_+ in the Hilbert space $\mathcal{H}_{\eta_+}$ may now be considered:

$$\eta_+ := \mathrm{e}^{-2\gamma \mathcal{L}_z}, \qquad \gamma \in R. \tag{6}$$

Metric of the form (6) has been considered previously in the study of a variety of pseudo-Hermitian quantum mechanical systems [2, 7, 8, 20]. The operators (x, y) and (p_x, p_y) are no more Hermitian in the Hilbert space $\mathcal{H}_{\eta_+}$. A new set of canonical conjugate operators that are Hermitian in the Hilbert space $\mathcal{H}_{\eta_+}$ may be introduced as follows:

$$\begin{aligned} &X = x \cosh w + \mathrm{i} y \sinh w, \qquad Y = -\mathrm{i} x \sinh w + y \cosh w, \qquad Z = z, \\ &P_X = p_x \cosh w + \mathrm{i} p_y \sinh w, \qquad P_Y = -\mathrm{i} p_x \sinh w + p_y \cosh w, \qquad P_Z = p_z, \\ &w \equiv \gamma + \mathrm{i}\xi, \qquad \xi \in R. \end{aligned} \tag{7}$$

The transformation matrix M,

$$M \equiv \begin{pmatrix} \cosh w & \mathrm{i} \sinh w \\ -\mathrm{i} \sinh w & \cosh w \end{pmatrix}, \tag{8}$$

relating (X, Y) to (x, y) and (P_X, P_Y) to (p_x, p_y), has appeared previously in the study of two-level pseudo-Hermitian quantum systems [2, 20]. Note that the length remains invariant under the transformation defined by equation (7), i.e. $R^2 \equiv X^2 + Y^2 + Z^2 = r^2 \equiv x^2 + y^2 + z^2$. The same is true for the total momentum square, $P^2 \equiv P_X^2 + P_Y^2 + P_Z^2 = p^2 \equiv p_x^2 + p_y^2 + p_z^2$. The operators (X, Y, P_X, P_Y) defined by equation (7) are not Hermitian with respect to the standard inner product for $\gamma \neq 0$. The angular momentum operators,

$$\begin{aligned} &L_X := Y P_Z - Z P_Y, \qquad &&L_Y := Z P_X - X P_Z, \\ &L_Z := X P_Y - Y P_X, \qquad &&L^2 := L_X^2 + L_Y^2 + L_Z^2, \end{aligned} \tag{9}$$

are related to $\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z$ and $\mathcal{L}^2 := \mathcal{L}_x^2 + \mathcal{L}_y^2 + \mathcal{L}_z^2$ through the equations

$$\begin{aligned} &L_X = \cosh w \mathcal{L}_x + \mathrm{i} \sinh w \mathcal{L}_y, \\ &L_Y = -\mathrm{i} \sinh w \mathcal{L}_x + \cosh w \mathcal{L}_y, \\ &L_Z = \mathcal{L}_z, \qquad L^2 = \mathcal{L}^2. \end{aligned} \tag{10}$$

For $\gamma \neq 0$, the operators L_X and L_Y are not Hermitian with respect to the standard inner product, but are Hermitian with respect to the modified inner product. The operators (Z, P_Z, L_Z, R, P, L^2) or equivalently $(z, p_z, \mathcal{L}_z, r, p, \mathcal{L}^2)$ are Hermitian in $\mathcal{H}_{\mathcal{D}}$ as well as in $\mathcal{H}_{\eta_+}$. The operators X, Y, P_X, P_Y, L_X, L_Y are Hermitian in $\mathcal{H}_{\eta_+}$ and the quasi-hermiticity [4] of these operators may be checked as follows:

$$\begin{aligned} &x = (U\rho) X (U\rho)^{-1}, \qquad y = (U\rho) Y (U\rho)^{-1}, \qquad p_{x,y} = (U\rho) P_{X,Y} (U\rho)^{-1}, \\ &\mathcal{L}_{x,y} = (U\rho) L_{X,Y} (U\rho)^{-1}, \qquad U := \mathrm{e}^{-\mathrm{i}\xi \mathcal{L}_z}, \qquad U^\dagger = U^{-1} = \mathrm{e}^{\mathrm{i}\xi \mathcal{L}_z}, \end{aligned} \tag{11}$$

where t^* denotes complex conjugation of t. The property of quasi-hermiticity of the operators $X, Y, P_{X,Y}, L_{X,Y}$ may be shown without the use of the unitary operator U. For example,

$$\begin{aligned} &\rho X \rho^{-1} = x \cos \xi - y \sin \xi, \\ &\rho Y \rho^{-1} = x \sin \xi + y \cos \xi. \end{aligned} \tag{12}$$

Similar relations between $P_{X,Y}(L_{X,Y})$ and $p_{x,y}(\mathcal{L}_{x,y})$ also exist. The unitary operation using U has been performed in equation (11) to rotate away insignificant terms in the expressions of equivalent Hermitian operators in $\mathcal{H}_D$. For $\gamma = 0$, η_+ reduces to the identity operator, and hence, both the Hilbert spaces become identical. If we further fix $\xi = 0$, (X, Y, Z, P_X, P_Y, P_Z) become identical to (x, y, z, p_x, p_y, p_z).

The metric operator η_+ is Hermitian. If the factor -2γ is replaced by a purely imaginary number $-\mathrm{i}\phi$, $\phi \in (0, 2\pi)$, it would correspond to a rotation by an angle ϕ around the z-axis. The operator η_+ given in equation (6) may thus be referred to as generating a 'complex rotation' around the z-axis by an amount equal to 2γ. The metric operator corresponding to a 'complex rotation' around the x- or y-axis or even around any arbitrary three-dimensional unit vector $\hat{n}$ may be constructed: $\hat{\eta}_+ := \mathrm{e}^{-2\gamma \hat{n}\cdot\vec{\mathcal{L}}}$, where $\vec{\mathcal{L}}$ is the three-dimensional angular momentum operator and the results of this paper may be generalized. In this paper, however, discussion is restricted to the metric η_+ given by equation (6), unless mentioned otherwise.

Suitable combinations of the operators X, Y, Z, P_X, P_Y, P_Z would result in a very large number of pseudo-Hermitian quantum systems, since these operators are pseudo-Hermitian by construction. A general non-relativistic pseudo-Hermitian quantum system in an external static electric and magnetic field is described by a Hamiltonian of the form [2, 17, 18]

$$H = \frac{1}{2m}(\vec{P} - e\vec{A}(X, Y, Z))^2 + V(X, Y, Z) + eA_0(X, Y, Z), \tag{13}$$

where the vector potential $\vec{A}(X, Y, Z)$, scalar potential $A_0(X, Y, Z)$ and potential $V(X, Y, Z)$ are real functions of their arguments. The constants m and e are the mass and the charge of the particle, respectively. The form of the minimal coupling to the gauge field in the Hamiltonian

H is determined by demanding $U(1)$ invariance [17, 18]. Subtleties involving the gauge transformation in pseudo-Hermitian quantum systems are discussed in [17, 18]. The reason for a straightforward and form-invariant extension of the minimal gauge coupling principle of a Hermitian theory to a pseudo (quasi)-Hermitian theory could be understood in a simple manner. The Hamiltonian H is Hermitian in $\mathcal{H}_{\eta_+}$ and non-Hermitian in $\mathcal{H}_D$, when expressed in terms of (x, y, z, p_x, p_y, p_z). However, it can be mapped to a Hermitian Hamiltonian h in $\mathcal{H}_D$ through the similarity transformation:

$$\begin{aligned} h &= (U\rho)H(U\rho)^{-1} \\ &= \frac{1}{2m}(\vec{p} - e\vec{A}(x, y, z))^2 + V(x, y, z) + eA_0(x, y, z). \end{aligned} \tag{14}$$

The minimal gauge coupling in h due to $U(1)$ invariance has the standard form. Thus, the form of the coupling to the gauge field in H is justified, if the standard minimal coupling principle due to $U(1)$ gauge invariance is to be maintained for the equivalent Hermitian Hamiltonian h. The Coulomb-gauge condition in $\mathcal{H}_D$ is $\vec{p} \cdot \vec{A}(x, y, z) = 0$, and in $\mathcal{H}_{\eta_+}$, it is

$$\vec{P} \cdot \vec{A}(X, Y, Z) = (U\rho)^{-1}(\vec{p} \cdot \vec{A}(x, y, z))(U\rho) = 0. \tag{15}$$

It may be noted that H and h are isospectral, since they are related to each other through a similarity transformation. However, the eigenfunctions are different. The electromagnetic transition rate between two given states is also identical [17, 18].

The spin degrees of freedom of a particle can also be included in the discussion of a non-relativistic pseudo-Hermitian quantum system. To this end, a positive-definite metric in the Hilbert space may be defined as the direct product of η_+ and the metric ζ_+ corresponding to the spin degrees of freedom,

$$\eta_+^{\text{Total}} := \eta_+ \otimes \zeta_+, \qquad \zeta_+ := \mathrm{e}^{-2\delta\hat{m}\cdot\vec{\mathcal{S}}}, \qquad \delta \in R, \tag{16}$$

where $\vec{\mathcal{S}}$ is the spin operator with components $\mathcal{S}_{x,y,z}$ which are Hermitian with respect to the standard inner product and $\hat{m}$ is a unit vector. The Hermitian spin operators $T_{X,Y,Z}$ with respect to the modified inner product $\langle\langle\cdot,\cdot\rangle\rangle_{\zeta_+} := \langle\cdot, \zeta_+\cdot\rangle$ may now be constructed using equation (4). Restricting the discussion to a simpler case where $\hat{m}$ corresponds to a unit vector along $\mathcal{S}_z$, the operators $T_{X,Y,Z}$ may be defined as

$$\begin{aligned} T_X &:= \cosh\beta\mathcal{S}_x + \mathrm{i}\sinh\beta\mathcal{S}_y, \\ T_Y &:= -\mathrm{i}\sinh\beta\mathcal{S}_x + \cosh\beta\mathcal{S}_y, \\ T_Z &:= \mathcal{S}_z, \qquad \beta \equiv \delta + \mathrm{i}\chi, \qquad \chi \in R. \end{aligned} \tag{17}$$

It may be noted that (T_Z, T^2) and $(\mathcal{S}_z, \mathcal{S}^2)$ are Hermitian with respect to both $\langle\cdot,\cdot\rangle$ and $\langle\langle\cdot,\cdot\rangle\rangle_{\zeta_+}$. The spin–orbit interaction of the form $H_{LS} = f(R)\vec{L}\cdot\vec{T}$, where $f(R)$ is a real function of R, is Hermitian with respect to the inner product $\langle\langle\cdot,\cdot\rangle\rangle_{\eta_+^{\text{Total}}}$. Thus, the Hamiltonian $\tilde{H} = H + H_{LS}$ or its variants involving both spatial and spin degrees of freedom are Hermitian in the Hilbert space $\mathcal{H}_{\eta_+^{\text{Total}}}$ that is endowed with the metric η_+^{Total}.

Several examples realizing the above formalism are considered in the next few sections. Examples in this paper are chosen based on their simplicity, physical relevance and, in some cases, exact solvability. The last criterion is very important in the following sense. The positive-definite metric in the Hilbert space cannot be calculated exactly for many of the pseudo-Hermitian quantum systems known so far. Perturbative and/or numerical methods are used to find approximate form of the metric. Accuracy of these methods may be checked by using an exactly solvable pseudo-Hermitian system. The physical relevance of the chosen quantum system is also very important, since experimental realization of the predictions emanating from pseudo-Hermitian/$\mathcal{PT}$-symmetric quantum mechanics is desirable.

3. Examples: single-particle system

In this section, examples of (i) a two-dimensional simple harmonic oscillator with complex angular frequencies, (ii) Stark effect in an external uniform complex electric field and (iii) Zeeman effect with non-Hermitian interaction are considered.

3.1. Simple harmonic oscillator

A $\mathcal{PT}$-symmetric oscillator in one dimension has been considered in the literature [11], where the Hermitian and the non-Hermitian Hamiltonian in $\mathcal{H}_D$ are related to each other through an imaginary shift of the coordinate. A two-dimensional simple harmonic oscillator with complex angular frequencies $\{\omega_1, \omega_2, \omega_3\}$ that admit entirely real spectra is presented below:

$$
\begin{aligned}
H &= \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{m}{2}\left(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 xy\right), \\
m\omega_1^2 &= k_1 \cos h^2 w - k_2 \sinh^2 w - \mathrm{i}k_3 \cosh w \sinh w, \\
m\omega_2^2 &= k_2 \cos h^2 w - k_1 \sinh^2 w + \mathrm{i}k_3 \cosh w \sinh w, \\
m\omega_3^2 &= 2\mathrm{i}(k_1 - k_2)\cosh w \sinh w + k_3(\cos h^2 w + \sinh^2 w),
\end{aligned}
\tag{18}
$$

where $\{k_1, k_2, k_3, m\} \in R$. The non-Hermitian Hamiltonian in $\mathcal{H}_D$ is related to a Hermitian Hamiltonian in $\mathcal{H}_D$ through a complex-hyperbolic transformation of the form (7). It is worth mentioning here that simple harmonic oscillators with complex angular frequencies appear in the description of electromagnetic pulse propagation in a free-electron laser [25]. Simple harmonic oscillators with complex angular frequencies have also been studied in the context of squeezed states [26], coherent states [27], tunneling phenomenon in non-Hermitian theory [28] and resonant states [29]. In general, the eigenvalues for the above cases are complex and the time evolution is non-unitary. Within the context of $\mathcal{PT}$-symmetric theory on a non-commutative space or with a deformed Heisenberg algebra, a simple harmonic oscillator with complex mass and complex angular frequency that admits real spectra within restricted regions of the parameter space has also been studied [30]. The harmonic oscillator Hamiltonian presented in this paper is on the standard Euclidean space and with real mass.

Although H is non-Hermitian in $\mathcal{H}_D$, it is Hermitian in $\mathcal{H}_{\eta_+}$. In particular, H can be rewritten as

$$
H = \frac{1}{2m}\left(P_X^2 + P_Y^2\right) + \frac{1}{2}(k_1 X^2 + k_2 Y^2 + k_3 XY). \tag{19}
$$

The quasi-hermiticity of H can also be shown by mapping it to $h := (U\rho)H(U\rho)^{-1}$:

$$
h = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}(k_1 x^2 + k_2 y^2 + k_3 xy), \tag{20}
$$

where the operators U and ρ are as defined below:

$$
U := \mathrm{e}^{-\mathrm{i}\xi \mathcal{L}_z}, \qquad \rho := \mathrm{e}^{-\gamma \mathcal{L}_z}. \tag{21}
$$

The Hamiltonian h is Hermitian in $\mathcal{H}_D$. The energy eigenvalues of h and hence, of H are determined for the range of the parameters $k_{1,2} > 0, 4k_1k_2 > k_3^2$ as follows:

$$
\begin{aligned}
E_{n_x,n_y} &= \left(n_x + \frac{1}{2}\right)\lambda_+ + \left(n_y + \frac{1}{2}\right)\lambda_-, \\
\lambda_\pm &\equiv \frac{1}{2\sqrt{m}}\left(k_1 + k_2 \pm \sqrt{k_3^2 + (k_1 - k_2)^2}\right)^{\frac{1}{2}}, \qquad n_x, n_y = 0, 1, 2, \ldots.
\end{aligned}
\tag{22}
$$

The corresponding eigenfunctions of H are

$$\Psi_{n_x,n_y}(u,v)=\mathrm{e}^{wL_z(u,v)}\psi_{n_x}(u)\psi_{n_y}(v),\qquad L_z(u,v)=-\mathrm{i}\left(u\frac{\partial}{\partial v}-v\frac{\partial}{\partial u}\right),\tag{23}$$

where $\psi_n(u)$ corresponds to the nth normalized eigenfunction of the standard one-dimensional simple harmonic oscillator and the coordinates (u, v) are obtained from (x, y) through a rotation on the two-dimensional plane by an angle θ:

$$\theta=\frac{1}{2}\tan^{-1}\frac{k_3}{k_1-k_2},\qquad k_1\neq k_2;\qquad \theta=\frac{\pi}{4},\qquad k_1=k_2.\tag{24}$$

For $\lambda_+ \neq \lambda_-$ (i.e. $k_1 \neq k_2$ and $k_3 \neq 0$), neither $\psi_{n_x}(u)\psi_{n_y}(v)$ are eigenfunctions of $L_z(u, v)$ nor a basis can be chosen in which simultaneous eigenfunctions of $L_z(u, v)$ and $h(u, v)$ can be constructed. The following identity can be shown using the properties of the Hermite polynomials:

$$\bar{L}_z[\psi_n(u)\psi_m(v)]=m\psi_{n+1}(u)\psi_{m-1}(v)-n\psi_{n-1}(u)\psi_{m+1}(v),\qquad \bar{L}_z\equiv \mathrm{i}L_z.\tag{25}$$

It may be noted that the ground state $\Psi_{0,0}(u, v) = \psi_0(u)\psi_0(v)$. However, the excited states are determined in terms of an infinite series and a general term in this series will contain an expression of the form

$$\begin{aligned}\bar{L}_z^k[\psi_n(u)\psi_m(v)]&=\sum_{i=0}^{\frac{k}{2}}[A_{ki}\psi_{n+2i}(u)\psi_{m-2i}(v)+B_{ki}\psi_{n-2i}(u)\psi_{m+2i}(v)],\quad \text{even } k\\&=\sum_{i=0}^{\frac{k-1}{2}}[A_{ki}\psi_{n+2i+1}(u)\psi_{m-2i-1}(v)+B_{ki}\psi_{n-2i-1}(u)\psi_{m+2i+1}(v)],\quad \text{odd } k,\end{aligned}\tag{26}$$

where the real constants A_{ki} and B_{ki} are determined in terms of n and m for fixed k and i. Thus, the wavefunction $\Psi_{n_x,n_y}(u, v)$ is expressed in terms of its arguments in a non-trivial way. It is worth mentioning that the explicit form of Ψ_{n_x,n_y} in terms of an infinite series is not required to calculate expectation values or matrix elements of any observables in $\mathcal{H}_{\eta_+}$. The form of Ψ_{n_x,n_y}, as given in equation (23), is sufficient for this purpose. The factor e^{wL_z} cancels out with a similar factor coming from the metric in the Hilbert space. In particular, $\langle\langle\Psi_{n'_x n'_y}|\hat{A}|\Psi_{n_x n_y}\rangle\rangle_{\eta_+} = \langle\psi_{n'_x}(u)\psi_{n'_y}(v)|\hat{A}|\psi_{n_x}(u)\psi_{n_y}(v)\rangle$. The orthonormal property of ψ_{n_x} together with equation (23) can be used to show that $\Psi_{n_x,n_y}(u, v)$ for all allowed values of the quantum numbers n_x and n_y constitute a complete set of orthonormal wavefunctions in $\mathcal{H}_{\eta_+}$.

3.2. Stark effect

A non-Hermitian Hamiltonian in $\mathcal{H}_D$ may be considered as

$$H=\frac{p^2}{2m}-\frac{e^2}{r}+e\mathcal{E}(x\cosh w+\mathrm{i}\,y\sinh w),\qquad \mathcal{E}\in R,\tag{27}$$

which has a similarity with the Hamiltonian describing the Hydrogen atom in an external uniform 'static complex electric field' $\vec{\mathcal{E}} = -\mathcal{E}(\cosh w\hat{i} + \hat{j}\,\mathrm{i}\sinh w)$, where $\hat{i}$, $\hat{j}$ correspond to the unit vectors along the x- and y-directions, respectively. It may be noted that although the magnitude of the external electric field is real, the y-component of the electric field is purely imaginary. This is the reason for identifying H in $\mathcal{H}_D$ as describing the Stark effect in a 'static complex electric field'. It is worth mentioning here that with the proper identification of a set of canonical operators, H in $\mathcal{H}_{\eta_+}$ can be identified as describing the Stark effect with a real electric field. The possibility of generating non-Hermitian interaction terms of H in $\mathcal{H}_D$ in the effective description of some hitherto unknown quantum mechanical system which is not subjected to any external electric field also exists. However, in the absence of any concrete proposal on how such non-Hermitian interaction could be realized in realistic physical systems, the Hamiltonian should be considered as hypothetical. Nevertheless, the related discussions may be useful to elucidate many technical issues related to pseudo-Hermitian quantum systems.

The Hamiltonian H can be rewritten as

$$H = \frac{P^2}{2m} - \frac{e^2}{R} + e\mathcal{E}X, \tag{28}$$

implying that it is Hermitian in $\mathcal{H}_{\eta_+}$. The equivalent Hermitian Hamiltonian in $\mathcal{H}_D$,

$$h := (U\rho)H(U\rho)^{-1} = \frac{p^2}{2m} - \frac{e^2}{r} + e\mathcal{E}x, \tag{29}$$

can be cast into the standard form,

$$\tilde{h} := \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\mathcal{L}_y} h \, \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\mathcal{L}_y} h = \frac{p^2}{2m} - \frac{e^2}{r} + e\mathcal{E}z, \tag{30}$$

by a $\frac{\pi}{2}$ rotation around the y-axis, where the operators U and ρ are as defined in equation (21). The operators h, $\tilde{h}$ and H are isospectral, since they are related to each other through similarity transformations. Following the discussions of [17, 18], the electromagnetic transition rate between any two states is also identical for h and H.

The perturbative analysis of $h(\tilde{h})$ is given in any standard text on quantum mechanics. A perturbative analysis of the pseudo-Hermitian H in equation (27) may also be carried out directly to obtain the known results. The Hamiltonian H can be rewritten in terms of the unperturbed Hamiltonian H_0 and the perturbation H' as

$$H = H_0 + H', \qquad H_0 = \frac{p^2}{2m} - \frac{e^2}{r}, \qquad H' = e\mathcal{E}\,(x\cosh w + \mathrm{i}\, y \sinh w)\,. \tag{31}$$

The unperturbed Hamiltonian H_0 is Hermitian in $\mathcal{H}_D$ and commutes with $\mathcal{L}^2$ and $\mathcal{L}_z$. A complete set of orthonormal eigenstates of H_0 with energy E_n in $\mathcal{H}_D$ are denoted as ψ_{nlm}, where n is the principal quantum number, l is the azimuthal quantum number and m is the magnetic quantum number. The principal quantum number n can take any values from the set of positive integers, $l = 0, 1, \ldots, n-1$ and $m = -l, -l+1, \ldots, l-1, l$. The states ψ_{nlm} are simultaneous eigenstates of H, $\mathcal{L}^2$ and $\mathcal{L}_z$. The perturbing Hamiltonian H' is non-Hermitian in $\mathcal{H}_D$ and, in general,

$$\langle\psi_{nlm}|H'|\psi_{n'l'm'}\rangle \neq \langle\psi_{n'l'm'}|H'|\psi_{nlm}\rangle^*, \tag{32}$$

which can be checked easily by using the following identities:

$$\begin{aligned}
\langle\psi_{nlm}|\frac{H'}{e\mathcal{E}}|\psi_{n'l'm'}\rangle &= \langle\psi_{nlm}|\rho^{-1}x\rho|\psi_{n'l'm'}\rangle = \mathrm{e}^{(m-m')\gamma}\langle\psi_{nlm}|x|\psi_{n'l'm'}\rangle,\\
\langle\psi_{n'l'm'}|\frac{H'}{e\mathcal{E}}|\psi_{nlm}\rangle &= \langle\psi_{n'l'm'}|\rho^{-1}x\rho|\psi_{nlm}\rangle = \mathrm{e}^{-(m-m')\gamma}\langle\psi_{n'l'm'}|x|\psi_{nlm}\rangle.
\end{aligned} \tag{33}$$

The matrix elements are identical either for $m = m'$ or in the Hermitian limit $\gamma = 0$. The first-order correction to the ground state vanishes identically, since $\langle\psi_{100}|H'|\psi_{100}\rangle = e\mathcal{E}\langle\psi_{100}|x|\psi_{100}\rangle = 0$. The second-order correction to the ground state and the first-order correction to the first excited state involve product of the matrix elements of the form

$$\begin{aligned}
\langle\psi_{nlm}|H'|\psi_{n'l'm'}\rangle\langle\psi_{n'l'm'}|H'|\psi_{nlm}\rangle &= (e\mathcal{E})^2\,\langle\psi_{nlm}|x|\psi_{n'l'm'}\rangle\langle\psi_{n'l'm'}|x|\psi_{nlm}\rangle\\
&= (e\mathcal{E})^2\,|\langle\psi_{nlm}|x|\psi_{n'l'm'}\rangle|^2,
\end{aligned} \tag{34}$$

which is real and its value is equivalent to the case when the perturbation is taken as $e\mathcal{E}x$. It is worth recalling at this point that in the perturbative analysis of the equivalent Hermitian Hamiltonian $h := \rho H \rho^{-1} = H_0 + e\mathcal{E}x$, the perturbing term is indeed given by $e\mathcal{E}x$. Thus, the known results are reproduced. The change in the energy E_n of the state ψ_{nlm} due to perturbative corrections at all orders in H' higher than the first involves products of matrix elements of the form

$$\langle\psi_{nlm}|H'|\psi_{n_k l_k m_k}\rangle\langle\psi_{n_k l_k m_k}|H'|\psi_{n_{k-1}l_{k-1}m_{k-1}}\rangle\cdots\langle\psi_{n_2 l_2 m_2}|H'|\psi_{n_1 l_1 m_1}\rangle\langle\psi_{n_1 l_1 m_1}|H'|\psi_{nlm}\rangle, \tag{35}$$

where $k \geqslant 1$ and all the intermediate states $\psi_{n_k,l_k,m_k}, \psi_{n_{k-1}l_{k-1}m_{k-1}}, \ldots, \psi_{n_1,l_1,m_1}$ are different from the state $\psi_{n,l,m}$. Using identities (33), it may now be checked that the expression in equation (35) is equivalent to the following:

$$(e\mathcal{E})^{k+1}\langle\psi_{nlm}|x|\psi_{n_k l_k m_k}\rangle\langle\psi_{n_k l_k m_k}|x|\psi_{n_{k-1}l_{k-1}m_{k-1}}\rangle\cdots\langle\psi_{n_2 l_2 m_2}|x|\psi_{n_1 l_1 m_1}\rangle\langle\psi_{n_1 l_1 m_1}|x|\psi_{nlm}\rangle. \tag{36}$$

Further, for the application of the degenerate perturbation theory for $n > 1$, the elements of the $n^2 \times n^2$ matrix M determining the secular equation is of the form

$$[M]_{ij} = a_{ij}\,\mathrm{e}^{\gamma_i - \gamma_j}, \qquad a_{ij} = a_{ji} \in R, \qquad \gamma_i \in R. \tag{37}$$

Any matrix of this type can be shown to be pseudo-symmetric, i.e. M is related to its transpose M^T though a similarity transformation, $M^T = \eta M \eta^{-1}$, with the similar matrix η being given by $[\eta]_{ij} = \mathrm{e}^{-2\gamma_i}\delta_{ij}$. Consequently, M can be transformed to a symmetric matrix $\mathcal{M}$ as $\mathcal{M} = \rho M \rho^{-1}$ with $\rho := \sqrt{M}$ and $[\mathcal{M}]_{ij} = a_{ij}$. If the degenerate perturbation theory is applied to h with $e\mathcal{E}x$ as the perturbation, the matrix determining the secular equation is precisely of the form $\mathcal{M}$. Thus, the perturbative analysis of H and h gives identical results at each order of the perturbation.

A comment is in order before the end of this section. The Hamiltonian H_0 is also Hermitian in the Hilbert space $\mathcal{H}_{\eta_+}$. A complete set of orthonormal states of the Hamiltonian H_0 with the energy E_n may be constructed in the Hilbert space $\mathcal{H}_{\eta_+}$ as $\phi_{nlm} = (U\rho)^{-1}\psi_{nlm}$. The perturbing term $H' = e\mathcal{E}X$ is also Hermitian in $\mathcal{H}_{\eta_+}$ and the states ϕ_{nlm} can be used to calculate perturbative corrections at different orders to the energy E_n. The corrections to the energy eigenvalues may be obtained by replacing the standard inner product $\langle\cdot,\cdot\rangle$ with the modified inner product $\langle\langle\cdot,\cdot\rangle\rangle_{\eta_+}$. The identity

$$\langle\langle\phi_{nlm}|X|\phi_{n'l'm'}\rangle\rangle_{\eta_+} = \langle\psi_{nlm}|x|\psi_{n'l'm'}\rangle \tag{38}$$

is useful in establishing one-to-one correspondence between the perturbative corrections of H and h at each order. For example, the expansion of the ground-state energy $\tilde{E}_1$ of H up to the second order is obtained as

$$\begin{aligned}\tilde{E}_1 &= E_1 + e\mathcal{E}\langle\langle\phi_{100}|X|\phi_{100}\rangle\rangle_{\eta_+} + (e\mathcal{E})^2\sum_{n(\neq 1),l,m}\frac{|\langle\langle\phi_{100}|X|\phi_{nlm}\rangle\rangle_{\eta_+}|^2}{E_1 - E_n}\\ &= E_1 + e\mathcal{E}\langle\psi_{100}|x|\psi_{100}\rangle + (e\mathcal{E})^2\sum_{n(\neq 1),l,m}\frac{|\langle\psi_{100}|x|\psi_{nlm}\rangle|^2}{E_1 - E_n}.\end{aligned} \tag{39}$$

The results of perturbative analysis of H either in $\mathcal{H}_D$ or in $\mathcal{H}_{\eta_+}$ would give identical results at each order of perturbation.

3.3. Zeeman effect

A Hermitian Hamiltonian in $\mathcal{H}_{\eta_+}$ describing the Zeeman effect may be constructed as follows:

$$H = \frac{P^2}{2m} - \frac{e^2}{R} + \frac{1}{2m^2R}\frac{\mathrm{d}V(R)}{\mathrm{d}R}\vec{L}\cdot\vec{T} + \frac{e}{2m}\vec{B}\cdot(\vec{L} + 2\vec{T}) + \frac{e^2}{8m}(\vec{B}\times\vec{R})^2, \tag{40}$$

where $\vec{B}$ is an external uniform magnetic field and $V(R)$ is a real function of its argument which can be chosen to be the Coulomb potential. Unlike the case of the Stark effect where the external electric field is complex, the magnetic field $\vec{B}$ describing the Zeeman effect is real both in $\mathcal{H}_D$ and in $\mathcal{H}_{\eta_+}$. It should be mentioned here that the vector potential producing the real magnetic field is not necessarily real in $\mathcal{H}_D$ for which the relevant position operators are (x, y, z). For example, the vector potential $\vec{A}$ with components $A_x = \frac{B}{2}(\mathrm{i}x \sinh w - y \cosh w)$, $A_y = \frac{B}{2}(x \cosh w + \mathrm{i}y \sinh w)$ and $A_z = 0$ produces real magnetic field along the z-direction. Both A_x and A_y have a real part and an imaginary part. The study of quantum mechanical systems with imaginary gauge potential has relevance in understanding different kinds of phase transitions [31]. Thus, the consideration of complex gauge potential is physically well motivated.

The Hamiltonian is non-Hermitian in $\mathcal{H}_D$, as can be seen by rewriting it in terms of the variables $x, y, z, \mathcal{L}_{x,y,z}, \mathcal{S}_{x,y,z}$. The Hamiltonian is Hermitian in both $\mathcal{H}_D$ and in $\mathcal{H}_{\eta_+}$ in the following two limits: (i) $\gamma = \delta = 0$ and (ii) $\gamma = \delta,\ \xi = \chi, \vec{B} = |\vec{B}|\hat{k}$, where $\hat{k}$ is a unit vector along the z-direction. The second limit is interesting in the following sense. The kinetic energy and the Coulomb potential terms are Hermitian both in $\mathcal{H}_D$ and in $\mathcal{H}_{\eta_+}$ without any restriction on the parameters. With the choice of $\gamma = \delta, \xi = \chi$, the spin–orbit interaction term of the Hamiltonian is Hermitian in $\mathcal{H}_D$ as well as in $\mathcal{H}_{\eta_+}$. The origin of non-hermiticity of the last two terms in $\mathcal{H}_D$ is physically well motivated through the introduction of imaginary gauge potential. These two terms also become Hermitian in $\mathcal{H}_D$ if the magnetic field is taken along the z-direction. Thus, the direction of the external magnetic field can be varied to switch over from Hermitian to non-Hermitian description of H in $\mathcal{H}_D$.

The equivalent Hermitian Hamiltonian $h := (U\rho)H(U\rho)^{-1}$ in $\mathcal{H}_D$ has the following form:

$$h = \frac{p^2}{2m} - \frac{e^2}{r} + \frac{1}{2m^2 r}\frac{\mathrm{d}V(r)}{\mathrm{d}r}\vec{\mathcal{L}}\cdot\vec{\mathcal{S}} + \frac{e}{2m}\vec{B}\cdot(\vec{\mathcal{L}} + 2\vec{\mathcal{S}}) + \frac{e^2}{8m}(\vec{B}\times\vec{r})^2. \tag{41}$$

Both H and h are isospectral and have an identical electromagnetic transition rate for two given states. However, the eigenfunctions are different from each other. The study on the eigenvalue problem of h is included in any standard book on quantum mechanics, and thus, no discussion in this regard is given in this paper. Further, a direct perturbative analysis of H either in the Hilbert space $\mathcal{H}_D$ or in $\mathcal{H}_{\eta_+}$ may be carried out following the discussions in the previous section on the Stark effect.

An experimental realization or verification of the predictions emanating from the study of pseudo-Hermitian/$\mathcal{PT}$-symmetric quantum mechanics is desirable. In this regard, the examples considered in this section may offer promising scenarios. If non-Hermitian interactions of the form described in this paper can be produced in the laboratory with γ being one of the externally controllable parameters, the transition rate between two allowed levels may be studied for $\gamma = 0$ and $\gamma \neq 0$. It may be recalled here that in the Hilbert space $\mathcal{H}_D$, $\gamma = 0$ and $\gamma \neq 0$ correspond to Hermitian and non-Hermitian Hamiltonians, respectively. According to the prediction of this paper, the transition rate between any two allowed levels would be independent of γ, if nature realizes pseudo-Hermitian/$\mathcal{PT}$-symmetric quantum systems.

4. Examples: many-body system

In this section, examples from many-body quantum systems are considered. General quadratic forms of N bosons (fermions) with non-Hermitian interactions, symmetric and asymmetric *XXZ* spin-chain Hamiltonian in an external uniform, complex magnetic field are considered in this section. A non-Hermitian version of the Haldane–Shastry spin chain and Lipkin–Meshkov–Glick model is also discussed.

4.1. Hamiltonian: general quadratic form of boson operators

The general quadratic form of N boson operators satisfying the commutation relations

$$\left[a_i, a_j^\dagger\right] = \delta_{ij}, \qquad [a_i, a_j] = 0 = \left[a_i^\dagger, a_j^\dagger\right], \qquad i, j = 1, 2, \ldots, N \tag{42}$$

appears in many diverse branches of physics. The operator $a_i^\dagger$ is the adjoint of a_i in the Hilbert space $\mathcal{H}_D$ and a_i $(a_i^\dagger)$ may be identified as the annihilation (creation) operator. A non-Hermitian general quadratic form involving these operators may be constructed as follows:

$$\begin{aligned} H &= \frac{1}{2}\sum_{i,j=1}^{N}\left[\alpha_{ij}\left(\mathrm{e}^{w_i-w_j}a_i^\dagger a_j + \mathrm{e}^{-(w_i-w_j)}a_j^\dagger a_i\right) + \beta_{ij}\left(\mathrm{e}^{-(w_i+w_j)}a_i a_j + \mathrm{e}^{w_i+w_j}a_i^\dagger a_j^\dagger\right)\right],\\ w_i &\equiv \gamma_i + \mathrm{i}\xi_i, \qquad \{\gamma_i, \xi_i, \alpha_{ij}, \beta_{ij}\} \in R, \quad \alpha_{ij} = \alpha_{ji}, \quad \beta_{ij} = \beta_{ji}. \end{aligned} \tag{43}$$

In a coordinate space realization of the algebra (42), H corresponds to a quantum system of N simple harmonic oscillators interacting with each other through non-Hermitian interaction in one dimension. Alternatively, the same Hamiltonian H can be identified as that of an N-dimensional oscillator with non-Hermitian interaction. The non-Hermitian interactions in equation (43) may be interpreted as arising due to an imaginary gauge potential. It may be noted that such imaginary gauge potentials are also relevant in the context of metal–insulator transitions or depinning of flux lines from extended defects in type-II superconductors [31]. In fact, with nearest-neighbor interaction only and $\beta_{ij} = 0\,\forall\, i, j$, H resembles the random-hopping model of [31]. For $N = 1$, H is known as the Swanson Hamiltonian [6] and has been studied extensively in the literature in the context of a $\mathcal{PT}$-symmetric and pseudo-Hermitian quantum system. It is worth mentioning here that a non-Hermitian $\mathcal{PT}$-symmetric two-mode Bose–Hubbard system has been studied in [22]. The Hamiltonian in [22] is different from the Hamiltonian in equation (43).

The claim of this paper is that the non-Hermitian H in equation (43) admits entirely real spectra with unitary time evolution for arbitrary N and within a fixed region in the parameter space. To substantiate this claim, the metric operator η_+ and the similar operator $\rho := \sqrt{\eta_+}$ may be introduced as

$$\eta_+ := \prod_{i=1}^{N}\mathrm{e}^{-2\gamma_i a_i^\dagger a_i}, \qquad \rho := \prod_{i=1}^{N}\mathrm{e}^{-\gamma_i a_i^\dagger a_i}. \tag{44}$$

A set of operators A_i and their adjoint $A_i^\dagger$ in the Hilbert space of $\mathcal{H}_{\eta_+}$ are introduced as follows:

$$A_i := \rho^{-1}a_i\rho = \mathrm{e}^{-\gamma_i}a_i, \qquad A_i^\dagger := \rho^{-1}a_i^\dagger\rho = \mathrm{e}^{\gamma_i}a_i^\dagger, \tag{45}$$

which satisfy the same algebra given by equation (42). A general eigenstate of the total boson number operator in the Hilbert space $\mathcal{H}_D$ may be introduced as $|n_1, \ldots, n_i, \ldots, n_N\rangle_{\mathcal{H}_D}$ with the following relations:

$$\begin{aligned} a_i|n_1, \ldots, n_i, \ldots, n_N\rangle_{\mathcal{H}_D} &= \sqrt{n_i}|n_1, \ldots, n_i - 1, \ldots, n_N\rangle_{\mathcal{H}_D},\\ a_i^\dagger|n_1, \ldots, n_i, \ldots, n_N\rangle_{\mathcal{H}_D} &= \sqrt{n_i + 1}|n_1, \ldots, n_i + 1, \ldots, n_N\rangle_{\mathcal{H}_D}. \end{aligned} \tag{46}$$

The corresponding state in the the Hilbert space $\mathcal{H}_{\eta_+}$ is determined as

$$|n_1, \ldots, n_i, \ldots, n_N\rangle_{\mathcal{H}_{\eta_+}} = \prod_{k=1}^{N}\mathrm{e}^{\gamma_k n_k}|n_1, \ldots, n_i, \ldots, n_N\rangle_{\mathcal{H}_D}, \tag{47}$$

with the action of $A_i(A_i^\dagger)$ on $|n_1,\ldots,n_i,\ldots,n_N\rangle_{\mathcal{H}_{\eta_+}}$ given by the following relations:

$$\begin{aligned} A_i|n_1,\ldots,n_i,\ldots,n_N\rangle_{\mathcal{H}_{\eta_+}} &= \sqrt{n_i}|n_1,\ldots,n_i-1,\ldots,n_N\rangle_{\mathcal{H}_{\eta_+}}, \\ A_i^\dagger|n_1,\ldots,n_i,\ldots,n_N\rangle_{\mathcal{H}_{\eta_+}} &= \sqrt{n_i+1}|n_1,\ldots,n_i+1,\ldots,n_N\rangle_{\mathcal{H}_{\eta_+}}. \end{aligned} \tag{48}$$

The states $|n_1,\ldots,n_i,\ldots,n_N\rangle_{\mathcal{H}_{\eta_+}}$ form a complete set of orthonormal states in $\mathcal{H}_{\eta_+}$, while $|n_1,\ldots,n_{i+1},\ldots,n_N\rangle_{\mathcal{H}_D}$ form a complete set of orthonormal states in $\mathcal{H}_D$.

The Hamiltonian H is Hermitian in $\mathcal{H}_{\eta_+}$ and this can be checked easily by rewriting it as

$$H = \frac{1}{2}\sum_{i,j=1}^{N}\left[\alpha_{ij}\left(\mathrm{e}^{\mathrm{i}(\xi_i-\xi_j)}A_i^\dagger A_j + \mathrm{e}^{-\mathrm{i}(\xi_i-\xi_j)}A_j^\dagger A_i\right) + \beta_{ij}\left(\mathrm{e}^{-\mathrm{i}(\xi_i+\xi_j)}A_iA_j + \mathrm{e}^{\mathrm{i}(\xi_i+\xi_j)}A_i^\dagger A_j^\dagger\right)\right]. \tag{49}$$

The Hamiltonian H can be mapped to a Hamiltonian h that is Hermitian in $\mathcal{H}_D$:

$$\begin{aligned} h &= \rho H \rho^{-1} \\ &= \frac{1}{2}\sum_{i,j=1}^{N}\left[\alpha_{ij}\left(\mathrm{e}^{\mathrm{i}(\xi_i-\xi_j)}a_i^\dagger a_j + \mathrm{e}^{-\mathrm{i}(\xi_i-\xi_j)}a_j^\dagger a_i\right) + \beta_{ij}\left(\mathrm{e}^{-\mathrm{i}(\xi_i+\xi_j)}a_ia_j + \mathrm{e}^{\mathrm{i}(\xi_i+\xi_j)}a_i^\dagger a_j^\dagger\right)\right], \end{aligned} \tag{50}$$

thereby showing the quasi-hermiticity of H. A further unitary transformation removes the phase factors from h. In particular,

$$\begin{aligned} U &:= \prod_{i=1}^{N}\mathrm{e}^{-\mathrm{i}\xi_i a_i^\dagger a_i} \\ \tilde{h} &= UhU^{-1} = \frac{1}{2}\sum_{i,j=1}^{N}\left[\alpha_{ij}\left(a_i^\dagger a_j + a_j^\dagger a_i\right) + \beta_{ij}\left(a_ia_j + a_i^\dagger a_j^\dagger\right)\right]. \end{aligned} \tag{51}$$

A general prescription to diagonalize (51) has been given in [32]. The basic steps involve the identification of the following $2N\times 2N$ matrices:

$$D = \begin{pmatrix}\hat{\alpha} & \hat{\beta}\\ \hat{\beta} & \hat{\alpha}\end{pmatrix}, \qquad \hat{I} = \begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix}, \qquad Q := \hat{I}D = \begin{pmatrix}\hat{\alpha} & -\hat{\beta}\\ \hat{\beta} & -\hat{\alpha}\end{pmatrix}, \tag{52}$$

where I is an $N\times N$ identity matrix, and $\hat{\alpha}$ and $\hat{\beta}$ are $N\times N$ matrices with the elements $[\hat{\alpha}]_{ij} = \alpha_{ij}$ and $[\hat{\beta}]_{ij} = \beta_{ij}$. It can be shown that the eigenvalues of the matrix Q are of the form $\Omega \in \{\Omega_1, \Omega_2, \ldots, \Omega_N, -\Omega_1, -\Omega_2, \ldots, -\Omega_N\}$. Further, if $\hat{u}_i$ is the eigenvector corresponding to the eigenvalue Ω_i of Q, then $-\Omega_i$ is another eigenvalue of Q with the eigenvector $\hat{J}\hat{u}_i$, where $\hat{J}$ is an anti-linear, idempotent operator that commutes with D and anti-commutes with $\hat{I}$ [32]. The energy eigenvalues of $\tilde{h}$ are

$$E_{\{n_i\}} = \sum_{i=1}^{N}\left(n_i + \frac{1}{2}\right)\Omega_i, \qquad \Omega_i > 0\,\forall\, i. \tag{53}$$

The stability criteria require a positive-definite Ω_i and, consequently, these results are valid only in those regions in the parameter space where D is strictly positive [32].

A comment is in order before the end of this section. Schwinger's oscillator model of angular momentum can be realized in terms of A_1, A_2 and their adjoint in $\mathcal{H}_{\eta_+}$. The following angular momentum operators satisfying the $SU(2)$ algebra may be defined:

$$\begin{aligned} \hat{J}_+ &:= A_1^\dagger A_2 = \mathrm{e}^{\gamma_1-\gamma_2}a_1^\dagger a_2, \\ \hat{J}_- &:= A_2^\dagger A_1 = \mathrm{e}^{-(\gamma_1-\gamma_2)}a_2^\dagger a_1, \\ \hat{J}_z &:= \tfrac{1}{2}\left(A_1^\dagger A_1 - A_2^\dagger A_2\right) = \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right), \end{aligned} \tag{54}$$

where $\hat{J}_+$ is the adjoint of $\hat{J}_-$ in $\mathcal{H}_{\eta_+}$. The operator $\hat{J}_z$ is Hermitian in $\mathcal{H}_D$ as well as in $\mathcal{H}_{\eta_+}$. The usual physical interpretation of Schwinger's oscillator model of angular momentum is equally applicable to the generators $\hat{J}_\pm, \hat{J}_z$ with the help of equations (46)–(48). Suitable combinations of these operators would result in a pseudo-Hermitian Hamiltonian with the metric η_+. One such example is the non-Hermitian deformation of the LMG model [24]:

$$H_{\mathrm{LMG}} = \omega_0 \hat{J}_z + \omega\left(\hat{J}_-^2 + \hat{J}_+^2\right), \tag{55}$$

where ω_0 and ω are real parameters. In the Hermitian limit, $\gamma_1 = \gamma_2 = 0$, the standard LMG model is reproduced which has been studied extensively in the literature [33]. For $\gamma_1 \neq 0 \neq \gamma_2$ H_{LMG} is isospectral with the standard LMG model.

4.2. Hamiltonian: general quadratic form of fermion operators

A set of canonical Fermi operators satisfying the anti-commutation relations

$$\left\{c_i, c_j^\dagger\right\} = 2\delta_{ij}, \qquad \{c_i, c_j\} = 0 = \left\{c_i^\dagger, c_j^\dagger\right\}, \qquad i, j = 1, 2, \ldots, N, \tag{56}$$

and a non-Hermitian Hamiltonian in terms of these operators may be introduced in $\mathcal{H}_D$ as follows:

$$\begin{gathered} H = \sum_{i,j=1}^{N} A_{ij} c_i^\dagger c_j \,\mathrm{e}^{w_i - w_j} + \frac{1}{2}\sum_{i,j=1}^{N} B_{ij}\left(c_i^\dagger c_j^\dagger \,\mathrm{e}^{(w_i+w_j)} + c_i c_j \mathrm{e}^{-(w_i+w_j)}\right), \\ A_{ij} = A_{ji} \in R, \qquad B_{ij} = -B_{ji} \in R. \end{gathered} \tag{57}$$

The complex parameters w_i's are defined in equation (43). The Hamiltonian H is Hermitian in $\mathcal{H}_{\eta_+}$ with the metric η_+ defined as

$$\eta_+ := \prod_{i=1}^{N} \mathrm{e}^{-2\gamma_i c_i^\dagger c_i}. \tag{58}$$

The Hamiltonian can be mapped to a Hermitian Hamiltonian h in $\mathcal{H}_D$ by using the similar operator $\rho := \prod_{i=1}^{N} \mathrm{e}^{-\gamma_i c_i^\dagger c_i}$ and a unitary operator $U := \prod_{i=1}^{N} \mathrm{e}^{-\mathrm{i}\xi_i c_i^\dagger c_i}$ as

$$h := (U\rho) H (U\rho)^{-1} = \sum_{i,j=1}^{N} A_{ij} c_i^\dagger c_j + \frac{1}{2}\sum_{i,j=1}^{N} B_{ij}\left(c_i^\dagger c_j^\dagger + c_i c_j\right). \tag{59}$$

The Hamiltonian h is exactly solvable and the diagonalization procedure is described in detail in [34]. For the nearest-neighbor interaction, $H(h)$ can be mapped to a solvable non-Hermitian (Hermitian) XY spin chain in $\mathcal{H}_D$ by using the Jordan–Wigner transformation [34].

The fermionic annihilation operators C_i and their adjoint $C_i^\dagger$ in $\mathcal{H}_{\eta_+}$ may be defined in terms of $c_i, c_i^\dagger$ as

$$C_i := \mathrm{e}^{-\gamma_i} c_i, \qquad C_i^\dagger := \mathrm{e}^{\gamma_i} c_i^\dagger, \tag{60}$$

which satisfy the basic canonical anti-commutation relations (56). A general eigenstate of the total fermion number operator in the Hilbert space $\mathcal{H}_D$, $|f_1, \ldots, f_i, \ldots, f_N\rangle_{\mathcal{H}_D}$, is related to the corresponding state $|f_1, \ldots, f_i, \ldots, f_N\rangle_{\mathcal{H}_{\eta_+}}$ in the Hilbert space $\mathcal{H}_{\eta_+}$ through the following relation:

$$|f_1, \ldots, f_i, \ldots, f_N\rangle_{\mathcal{H}_{\eta_+}} = \prod_{k=1}^{N} \mathrm{e}^{\gamma_k f_k} |f_1, \ldots, f_i, \ldots, f_N\rangle_{\mathcal{H}_D}, \qquad f_i = 0, 1\,\forall\, i. \tag{61}$$

The 2^N states $|f_1,\ldots,f_i,\ldots,f_N\rangle_{\mathcal{H}_{\eta_+}}$ form a complete set of orthonormal states in $\mathcal{H}_{\eta_+}$, while $|f_1,\ldots,f_i,\ldots,f_N\rangle_{\mathcal{H}_D}$ constitute a complete set of orthonormal states in $\mathcal{H}_D$. The action of $C_i\,(C_i^\dagger)$ on $|f_1,\ldots,f_i,\ldots,f_N\rangle_{\mathcal{H}_{\eta_+}}$ is identical to that of c_i $(c_i^\dagger)$ on $|f_1,\ldots,f_i,\ldots,f_N\rangle_{\mathcal{H}_D}$. In particular,

$$\begin{aligned} C_i|f_1,\ldots,f_i,\ldots,f_N\rangle_{\mathcal{H}_{\eta_+}} &= 0, \qquad \text{if} \quad f_i = 0 \\ &= |f_1,\ldots,0,\ldots,f_N\rangle_{\mathcal{H}_{\eta_+}}, \qquad \text{if} \quad f_i = 1, \\ C_i^\dagger|f_1,\ldots,f_i,\ldots,f_N\rangle_{\mathcal{H}_{\eta_+}} &= 0, \qquad \text{if} \quad f_i = 1 \\ &= |f_1,\ldots,1,\ldots,f_N\rangle_{\mathcal{H}_{\eta_+}}, \qquad \text{if} \quad f_i = 0. \end{aligned} \tag{62}$$

Suitable combinations of the operators C_i and $C_i^\dagger$ would give rise to a very large number of pseudo-Hermitian quantum systems that go beyond the general quadratic form of fermionic oscillators. Further, the definitions of $C_i, C_i^\dagger$ could be generalized easily to accommodate a pseudo-Hermitian description of the Hubbard model, t–j model, etc. As in the case of bosonic oscillators, the $SU(2)$ generators can be realized in terms of pseudo-Hermitian fermion operators.

4.3. XXZ spin chain

The study of non-Hermitian spin chains has a long history. It is a well-known fact that non-Hermitian quantum spin chains correspond to two-dimensional classical systems with positive Boltzmann weights. The non-Hermitian *XY* and *XXZ* spin-chain Hamiltonians with Dzyaloshinsky–Moriya interaction commute with the transfer matrix of the six-vertex model in the presence of an electric field [35] and the integrable chiral Potts model in the most general case leads to a non-Hermitian quantum Hamiltonian [36, 37]. Non-Hermitian asymmetric XXZ spin chains related to diffusion models have been studied extensively in non-equilibrium statistical mechanics [38]. Further, a non-Hermitian quantum Ising spin chain in one dimension [39] is known to be related to the popular Yang–Lee model [40] that aptly describes ordinary second-order phase transitions. The non-hermiticity of the spin chain arises due to the inclusion of an external complex magnetic field and an analysis based on minimal conformal field theory is available [41]. Within the context of $\mathcal{PT}$-symmetric theory, non-Hermitian spin chains have been studied in [8, 9].

The pseudo-Hermitian spin operators $T_{x,y,z}$ and the metric operator ζ_+, as given in equations (16) and (17), may be generalized appropriately to introduce a pseudo-Hermitian *XXZ* spin-chain Hamiltonian. One such simple generalization is to consider the spin operators $T_i^{x,y,z}$

$$\begin{aligned} T_i^x &:= \cosh w_i \mathcal{S}_i^x + \mathrm{i} \sinh w_i \mathcal{S}_i^y, \\ T_i^y &:= -\mathrm{i} \sinh w_i \mathcal{S}_i^x + \cosh w_i \mathcal{S}_i^y, \\ T_i^z &:= \mathcal{S}_i^z, \end{aligned} \tag{63}$$

which are Hermitian in the Hilbert space $\mathcal{H}_{\zeta_+}$ with the positive-definite metric ζ_+ defined as

$$\zeta_+ := \prod_{i=1}^N \mathrm{e}^{-2\gamma_i T_i^z}. \tag{64}$$

The operators $\mathcal{S}_i^{x,y,z}$ are Hermitian in the Hilbert space $\mathcal{H}_D$ with the standard inner product. An asymmetric *XXZ* spin chain in an external complex magnetic field may now be constructed that is manifestly non-Hermitian in $\mathcal{H}_D$

$$H_A = \sum_{i=1}^{N-1} \left[\Gamma \left(\mathrm{e}^{w_i - w_{i+1}} \mathcal{S}_i^+ \mathcal{S}_{i+1}^- + \mathrm{e}^{-(w_i - w_{i+1})} \mathcal{S}_i^- \mathcal{S}_{i+1}^+ \right) + \Delta \mathcal{S}_i^z \mathcal{S}_{i+1}^z \right.$$
$$\left. + \left(A_i \cosh w_i - \mathrm{i} B_i \sinh w_i\right) \mathcal{S}_i^x + \left(B_i \cosh w_i + \mathrm{i} A_i \sinh w_i\right) \mathcal{S}_i^y + C_i \mathcal{S}_i^z \right], \tag{65}$$

where $\mathcal{S}_i^{\pm} := \mathcal{S}_i^x \pm \mathrm{i}\mathcal{S}_i^y$, $\{\Gamma, \Delta, A_i, B_i, C_i\} \in R$ and w_i are as defined in equation (43). The non-Hermitian interaction in H_A may be interpreted as arising due to an imaginary vector potential as in the case of the Bose system described before. In fact, with a hard-core boson representation, H_A can be mapped to a nearest-neighbor version of H in equation (43).

The Hamiltonian H_A can be mapped to a Hermitian Hamiltonian in $\mathcal{H}_D$

$$h := U\left(\zeta_+^{\frac{1}{2}} H_A \zeta_+^{-\frac{1}{2}}\right) U^{-1}$$
$$= \sum_{i=1}^{N-1} \left[\Gamma \left(\mathcal{S}_i^x \mathcal{S}_{i+1}^x + \mathcal{S}_i^y \mathcal{S}_{i+1}^y \right) + \Delta \mathcal{S}_i^z \mathcal{S}_{i+1}^z + A_i \mathcal{S}_i^x + B_i \mathcal{S}_i^y + C_i \mathcal{S}_i^z \right], \qquad U := \prod_{i=1}^{N} \mathrm{e}^{-\mathrm{i}\chi_i \mathcal{S}_i^z}, \tag{66}$$

implying that both H_A and h have entirely real spectra. The asymmetric XXZ spin-chain Hamiltonian H_A is Hermitian in $\mathcal{H}_{\eta_+}$ and this may be checked easily by rewriting H_A as

$$H_A = \sum_{i=1}^{N-1} \left[\Gamma \left(T_i^+ T_{i+1}^- + T_i^- T_{i+1}^+ \right) + \Delta T_i^z T_{i+1}^z + A_i T_i^x + B_i T_i^y + C_i T_i^z \right], \tag{67}$$

where $T_i^{\pm} := T_i^x \pm \mathrm{i} T_i^y$. Thus, the time evolution of H_A is unitary in $\mathcal{H}_{\eta_+}$.

A few comments are in order at this point.

(i) Several variants of the asymmetric *XXZ* Hamiltonian (65) have been studied in the literature [38] in the context of two-species reaction–diffusion processes and Kardar–Parisi–Zhang-type growth phenomenon. A typical choice for w_k in these models is

$$\gamma_k = \gamma - (k-1)\phi, \qquad \xi_k = \xi \; \forall\, k, \qquad \{\gamma, \xi, \phi\} \in R, \tag{68}$$

leading to a site-independent global phase factor $\mathrm{e}^{\pm\phi}$ in lieu of $\mathrm{e}^{\pm(w_i - w_{i+1})}$. The transformation that maps a non-Hermitian asymmetric XXZ Hamiltonian to a Hermitian Hamiltonian is also known in the literature [38]. This transformation is generally used to show the reality of the entire spectra. However, with the standard inner product in the Hilbert space $\mathcal{H}_D$, negative norm states exist. Consequently, in spite of having an entirely real spectra, the time evolution of H_A in $\mathcal{H}_D$ is not unitary and dissipative processes can thus be stimulated. The pseudo-hermiticity of H_A has not been noted previously. The time evolution of H_A in $\mathcal{H}_{\eta_+}$ is unitary. Thus, with the discovery of the pseudo-hermiticity of H_A, it may be used to describe unitary time evolution in $\mathcal{H}_{\eta_+}$. At a purely formal level, it might seem to be a matter of choice to describe either unitary or non-unitary time evolution by fixing an appropriate metric in the Hilbert space. However, an experimental realization of any one of these systems may give a definite answer on whether nature realizes pseudo-Hermitian quantum systems or not.

(ii) The symmetric XXZ spin-chain Hamiltonian in an external complex magnetic field may be constructed by choosing $w_i \equiv w \equiv \gamma + \mathrm{i}\chi \; \forall\, i$, $\{\gamma, \chi\} \in R$ in equation (65):

$$H_S = \sum_{i=1}^{N-1} \left[\Gamma \left(\mathcal{S}_i^x \mathcal{S}_{i+1}^x + \mathcal{S}_i^y \mathcal{S}_{i+1}^y \right) + \Delta \mathcal{S}_i^z \mathcal{S}_{i+1}^z + \left(A_i \cosh w - \mathrm{i} B_i \sinh w\right) \mathcal{S}_i^x \right.$$
$$\left. + \left(B_i \cosh w + \mathrm{i} A_i \sinh w\right) \mathcal{S}_i^y + C_i \mathcal{S}_i^z \right], \tag{69}$$

which is non-Hermitian in $\mathcal{H}_D$, but Hermitian in $\mathcal{H}_{\zeta_+}$. The equivalent Hermitian Hamiltonian $h := U\big(\zeta_+^{\frac{1}{2}} H_S \zeta_+^{-\frac{1}{2}}\big)U^{-1}$ in $\mathcal{H}_D$ to H_S is still given by equation (66).

The Hamiltonian h has several integrable limits. Consequently, H_A and H_S are also integrable in these limits with entirely real spectra and unitary time evolution. For example, h reduces to a transverse-field Ising model for $\Gamma = B_i = C_i = 0$, $A_i = A\,\forall\, i$ and both h and H_S have been studied in some detail [8] for this limiting case. For $\Delta = 0$, $A_i = 0$, $B_i = 0\,\forall\, i$, h reduces to an XX model in a transverse magnetic field and is exactly solvable [34, 42]. Although H_S is Hermitian in $\mathcal{H}_D$ for this choice of the parameters, H_A is non-Hermitian. Thus, the non-Hermitian H_A is exactly solvable and has an equivalent description in terms of a Hermitian XX model in an external magnetic field. For the following choice of the parameters,

$$\begin{aligned} &\Gamma = 1, \quad \Delta = \cosh q, \quad C_1 = -C_N = -\sinh q, \quad A_i = B_i = 0\,\forall\, i; \\ &C_i = 0, \quad i = 2, 3, \ldots, N-1, \end{aligned} \tag{70}$$

$h - \Delta$ reduces to an $SU_q(2)$ invariant [43] integrable [44] spin chain Hamiltonian. The XXZ spin chain with Sl_2 loop symmetry [45] may also be obtained as a limiting case. The corresponding non-Hermitian Hamiltonian H_A is also integrable and allows a unitary description.

(iii) Only the spin chains with nearest-neighbor interactions are presented in this paper. A large number of pseudo-Hermitian spin chains with no restriction on the type of interactions (i.e. nearest-neighbor, next-nearest-neighbor etc) may be constructed by the use of operators $T_i^{x,y,z}$. For example, a non-Hermitian version of the celebrated Haldane–Shastry spin chain [23] may be constructed as follows:

$$H = \pm \sum_{i<j} \frac{\vec{T}_i \cdot \vec{T}_j}{2\sin^2 \frac{\pi}{N}(i-j)}, \tag{71}$$

where H is Hermitian in $\mathcal{H}_{\eta_+}$ and non-Hermitian in $\mathcal{H}_D$. The equivalent Hermitian Hamiltonian in $\mathcal{H}_D$ may be obtained as

$$h := (U\rho)\, H (U\rho)^{-1} = \pm \sum_{i<j} \frac{\vec{\mathcal{S}}_i \cdot \vec{\mathcal{S}}_j}{2\sin^2 \frac{\pi}{N}(i-j)}, \tag{72}$$

implying that h and H are isospectral, where U is as defined in equation (66). It may be noted that, in general, eigenstates of h and H are different. However, with proper identification of physical observables in $\mathcal{H}_{\eta_+}$ through equation (4), different correlation functions of the quantum systems governed by H and h are identical.

5. Conclusions and discussions

A class of pseudo-Hermitian quantum systems with a pre-determined metric in the Hilbert space has been presented. These quantum systems admit entirely real spectra. Moreover, the time evolution is unitary with the use of the modified inner product in the Hilbert space. The general approach that has been used in the construction of these quantum systems is the following. The basic canonical commutation relations defining these systems have been realized in terms of operators that are non-Hermitian with respect to the Dirac-hermiticity condition, but are Hermitian with respect to the modified inner product in the Hilbert space involving the pre-determined metric. Consequently, appropriate combinations of these operators result in a very large number of pseudo-Hermitian quantum systems. The examples considered in this paper include higher-dimensional simple harmonic oscillators with complex angular frequencies, Stark effect with complex electric field, Zeeman effect with non-Hermitian interaction, non-Hermitian general quadratic form of N boson (fermion) operators, XXZ spin chains with complex magnetic field, a non-Hermitian version of the Haldane–Shastry spin chain and Lipkin–Meshkov–Glick model.

The results presented in this paper are purely mathematical. An experimental realization or verification of the predictions emanating from the study of pseudo-Hermitian/$\mathcal{PT}$-symmetric quantum mechanics is desirable. Although a concrete proposal on how non-Hermitian interaction of the form described in this paper could be realized experimentally is lacking, it is worth mentioning possible signatures in support/violation of $\mathcal{PT}$-symmetric/pseudo-Hermitian quantum physics, even within hypothetical setups. In this regard, the examples considered in section 3 and time evolution of the asymmetric *XXZ* Hamiltonian may be promising scenarios. For example, if the non-Hermitian interaction of the form described in section 3 can be produced in the laboratory with γ being one of the externally controllable parameters, the transition rate between two allowed levels may be studied for $\gamma = 0$ and $\gamma \neq 0$. It may be recalled here that in the Hilbert space $\mathcal{H}_D$, $\gamma = 0$ and $\gamma \neq 0$ correspond to Hermitian and non-Hermitian Hamiltonians, respectively. According to the prediction of this paper, *the transition rate between any two allowed levels would be independent of* γ, *if nature realizes pseudo-Hermitian*/$\mathcal{PT}$*-symmetric quantum systems*.

In a similar way, the time evolution of H_A in $\mathcal{H}_D$ is expected to be non-unitary, while it is unitary in $\mathcal{H}_{\eta_+}$. At a purely formal level, it might seem to be a matter of choice to describe either unitary or non-unitary time evolution by fixing an appropriate metric in the Hilbert space. However, an experimental realization of any one of these systems related to reaction–diffusion processes and the Kardar–Parisi–Zhang-type growth phenomenon may give a definite answer on whether nature realizes pseudo-Hermitian quantum systems or not and whether or not a more general positive-definite metric in the Hilbert space than the one prescribed by Dirac is allowed. Any experimental result indicating the independence of different types of correlation functions on γ [8] would garner support in favor of pseudo-Hermitian/$\mathcal{PT}$-symmetric quantum mechanics.

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